

# Total convergence or general divergence in Small Divisors

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**Abstract.** *We study generic holomorphic families of dynamical systems presenting problems of small divisors with fixed arithmetic. We prove that we have convergence for all parameter values or divergence everywhere except for an exceptional set of zero  $\Gamma$ -capacity. We illustrate this general principle in different problems of small divisors. As an application we obtain new richer families of non-linearizable examples in the Siegel problem when Bruno condition is violated, generalizing previous results of Yoccoz and the author.*

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## 1) Introduction.

In this article we study generic (polynomial) holomorphic families of dynamical systems presenting problems of small divisors with fixed arithmetic. The principle our theorems illustrate is the following :

**We have total convergence for all parameter values or general divergence except maybe for a very small exceptional set of parameter values.**

The germinal idea can be traced back to Y. Ilyashenko where in [Il] he studies divergence in problems of small divisors from divergence of the homological (or linearized) equation. Ilyashenko's paper contains a remarkable idea. We find there, for the first time in Small Divisors, the study of linear deformation of the system and the use of the polynomial dependence of the new formal linearizations. A similar idea, but not quite in the same problem, was used by H. Poincaré to show that linear deformations of completely integrable hamiltonians are not generally completely integrable with analytic first integrals depending analytically on the parameter ([Poi] volume I chapter V). It is worth noting that this is the key preliminary step in his difficult proof of the non existence of non trivial local analytic first integrals in the three body problem.

Such a linear deformation has been fruitfully used by J.-C. Yoccoz. He proves that in the Siegel problem the quadratic polynomial is the worst linearizable holomorphic germ ([Yo] p. 58). The only ingredient in this proof that is not in Ilyashenko's one is the classical Douady-Hubbard straightening theorem for polynomial-like mappings. Yoccoz simplifies Ilyashenko's argument replacing Nadirashvili's lemma by the maximum principle.

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He loses in that way the strength of the original approach, in particular the potential theoretic aspects. Non-linear polynomial perturbations were used by the author in [PM1] to generalize Yoccoz's result to higher degree polynomials.

In this article we clarify and strength the role played by potential theory in parameter space. A key point is the observation that Nadirashvili's lemma can be improved by using Bernstein's lemma in approximation theory. In that way we do make precise the thinnest notion for the exceptional set. In parameter space the exceptional set has  $\Gamma$ -capacity 0 (for the definition of  $\Gamma$ -capacity see [Ro] section 2.2). The intersection of the exceptional set with any complex line is full (the whole line) or a polar set (any set of  $\Gamma$ -capacity 0 has this property, see [Ro] Lemma 2.2.8 p.92).

As far as the author knows, the first person who studied small divisors problems using ingredients from potential theory in parameter space is M. Herman (see [He1] and [He2]).

The techniques in this article are applicable to virtually any holomorphic problem in small divisors where the dependence on parameters of the coefficients of the divergent series are polynomial (as we will see this happens in most of the problems). We have selected a few illustrative ones guided mainly by our personal taste. We only consider here polynomial families. The same proof can be extended easily for more general holomorphic families (see remark 4).

It is surprising that the idea and the results of this article have been overlooked so far.

### Linearization.

**Theorem 1.** *Let  $n, m \geq 1$  and  $d \geq 0$ . For a multi-index  $i = (i_1, \dots, i_m) \in \mathbf{N}^m$  with  $0 \leq i_1 + \dots + i_m \leq d$ , let  $f_i$  be a germ of holomorphic map*

$$f_i : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$$

*with valuation larger or equal to 2 (i.e.  $f_i(z) = \mathcal{O}(z^2)$ ).*

*For  $t = (t_1, \dots, t_m) \in \mathbf{C}^m$  we consider the holomorphic family of germs of holomorphic diffeomorphisms,  $z \in \mathbf{C}^n$ ,*

$$f_t(z) = Az + \sum_{\substack{i=(i_1, \dots, i_m) \\ 0 \leq i_1 + \dots + i_m \leq d}} t^i f_i(z)$$

*where  $A \in GL_n(\mathbf{C})$  is a fixed linear map,  $A = D_0 f$ , with non-resonant eigenvalues.*

*Then all maps  $f_t$ ,  $t \in \mathbf{C}^m$  are formally linearizable, i.e. there exists a unique formal map  $h_t$  with  $h_t(0) = 0$  and  $D_0 h_t = I$  such that the formal equation*

$$h_t \circ f_t = A \circ h_t$$

*is satisfied.*

*We have the following dichotomy:*

1) The holomorphic family  $(f_t)_{t \in \mathbf{C}^m}$  is holomorphically linearizable, that is for all  $t \in \mathbf{C}^m$ ,  $h_t$  defines a germ of holomorphic diffeomorphism. Moreover, the radius  $R(t)$  of convergence of the linearization  $h_t$  is bounded from below on compact sets and, more precisely, for some  $C_0 > 0$ ,

$$R(t) \geq \frac{C_0}{1 + ||t||} .$$

2) Except for an exceptional set  $E \subset \mathbf{C}^m$  of  $\Gamma$ -capacity 0 of values of  $t$ ,  $f_t$  not holomorphically linearizable.

**Remarks:**

1) We remind that the eigenvalues  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  of  $A$ , counted with multiplicity, are non-resonant if

$$\lambda_i - \lambda_1^{i_1} \dots \lambda_n^{i_n} \neq 0$$

for all  $(i_1, \dots, i_n) \in \mathbf{N}^n$  with  $i_1 + \dots + i_n \geq 2$ . We give later a theorem for holomorphic germs with resonant linear parts.

2) The linear part  $A$  is in the Poincaré domain if

$$\min(\max_i |\lambda_i|, \max_i |\lambda_i^{-1}|) < 1 .$$

In that case it is well known that we are always in case (1). Otherwise the linear part of  $A$  belongs to the Siegel domain.

3) In general the exceptional set  $E \subset \mathbf{C}^m$  is not empty. For example if  $f_0 = 0$  and  $0 \in W$ , then  $0 \in E$  when we are in the second case.

4) With the same type of proof, one can prove the result for holomorphic families of the form

$$f_t(z) = Az + \sum_{\substack{i=(i_1, \dots, i_m) \\ 0 \leq i_1 + \dots + i_m}} t^i f_i(z)$$

where the holomorphic germs  $(f_i(z))$  have valuations such that  $\text{val} f_i \geq \varepsilon_0 |i|$  for some  $\varepsilon_0 > 0$ .

Some illustrative corollaries follow now. For  $n = 1$  and the special case of entire functions we have directly from theorem 1:

**Corollary 1.** *Let  $(f_t)_{t \in \mathbf{C}^m}$  be a finite dimensional holomorphic family of entire functions as above with*

$$f'_t(0) = \lambda$$

where  $\lambda = e^{2\pi i \alpha}$  with  $\alpha \in \mathbf{R} - \mathbf{Q}$ .

*Then the family is linearizable or, except for an exceptional polar set  $E \subset \mathbf{C}$  of values of  $t$ , all  $f_t$  are non-linearizable.*

Assuming that the family contains a non-linearizable structurally stable polynomial (for example a quadratic polynomial) we can break the dichotomy. This just follows from

the observation that in a neighborhood of this polynomial all elements of the family are quasi-conformally conjugated (by Douady-Hubbard straightening theorem), thus they are linearizable or not simultaneously.

**Corollary 2.** *Let  $(f_t)_{t \in \mathbf{C}^m}$  be a finite dimensional holomorphic family of entire functions as above with*

$$f'_t(0) = \lambda$$

*where  $\lambda = e^{2\pi i \alpha}$  with  $\alpha \in \mathbf{R} - \mathbf{Q}$ . We assume that for some value  $t_0$   $f_{t_0}$  is a structurally stable polynomial in the space of polynomials with fixed point at 0 and multiplier  $\lambda$ .*

*Then if  $\alpha$  is not a Bruno number almost all entire functions  $f_t$ , except maybe for an exceptional polar set  $E \subset \mathbf{C}$  of values of  $t$ , are not linearizable.*

When  $\alpha \in \mathbf{R} - \mathbf{Q}$  is not a Bruno number, no examples were known of non-linearizable entire functions not quasi-conformally conjugated to polynomials in a neighborhood of 0. This was due to the shortcomings of Yoccoz maximum principle approach [Yo].

A particular case of this corollary is the theorem proved in [PM1] about polynomial germs. The author showed, generalizing Yoccoz result for the quadratic polynomial, that if  $\alpha$  is not a Bruno number, in the space

$$\mathcal{P}_{\lambda,d} = \{P(z) = \lambda z + a_2 z^2 + \dots + a_d z^d; (a_2, \dots, a_d) \in \mathbf{C}^{d-1}\}$$

the polynomials that are not structurally stable (this is an open dense set whose complement has  $\Gamma$ -capacity 0) are not linearizable.

It is worth mentioning that the question to know if the exceptional set  $E_{\lambda,d}$  is empty for the polynomial family  $\mathcal{P}_{\lambda,d}$  when  $\alpha \in \mathbf{R} - \mathbf{Q}$  is not a Bruno number, is still open, even for the cubic family:

$$P_b(z) = \lambda z + b z^2 + z^3$$

Contrary to unanimous belief, the author will not be surprised that  $E_{\lambda,d}$  is not empty for appropriate values of  $\lambda$  and  $d$ . For Liouville numbers  $\alpha$  with extremely good rational approximations, by an argument of Cremer (see [Cr] and [PM1]),  $E_{\lambda,d}$  is known to be empty.

To illustrate the strength of the precedent theorem, we present the following variations.

**Corollary 3.** *Let  $\alpha \in \mathbf{R} - \mathbf{Q}$  be not Bruno.*

*1) Let  $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$  be non-linearizable. Any polynomial family  $(f_t)_{t \in \mathbf{C}}$  as above containing  $f$  has all of its members  $f_t$  non-linearizable except for an exceptional polar set of parameters  $t$ .*

*2) For an arbitrary holomorphic germ  $\varphi(z) = \mathcal{O}(z^2)$  and for almost all values  $t \in \mathbf{C}$  except a polar set  $E$ , we have that*

$$f_t(z) = e^{2\pi i \alpha} z + z^2 + t\varphi(z)$$

*is not linearizable.*

*3) Let*

$$f(z) = e^{2\pi i \alpha} z + \sum_{n \geq 2} f_n z^n$$

be an arbitrary entire function. Keeping all coefficients fixed except  $f_2$ , there is a polar set  $E$  such that if  $f_2 \in \mathbf{C} - E$ , then  $f$  is not linearizable.

Also, we have the same type of results for rational functions.

**Corollary 4.** *Let*

$$\mathcal{R}_{\lambda,d} = \{f \in \mathbf{C}(z); f(0) = 0; f'(0) = \lambda; d^0 f = d\}$$

When  $\alpha \in \mathbf{R} - \mathbf{Q}$  is not a Bruno number, except for an exceptional set, all rational functions in  $\mathcal{R}_{\lambda,d}$  are not linearizable.

The corollaries presented here are by no means restricted to dimension 1. Just one example of new result.

**Corollary 5.** *We consider the space  $\mathcal{P}_{A,d}$  of polynomial germs of holomorphic diffeomorphisms with non-resonant linear part  $A$  of total degree  $d$ . The existence of one non-linearizable example forces all the others except an exceptional set of  $\Gamma$ -capacity 0 to be non-linearizable. This happens for instance when one eigenvalue of  $A$  does not satisfy Bruno's condition.*

We can prove also a version of theorem 1 for resonant linear parts  $A$  which has an independent interest (for example when applied to symplectic holomorphic mappings). When the linear part is resonant, the linearization is not uniquely determined. Nevertheless, given a polynomial family  $(f_t)$  as in theorem 1 whose elements are all formally linearizable, there always exist a canonical family of linearizations  $(h_t)$  whose coefficients depend polynomially on  $t$  (see [PM3]). The complete treatment of this situation requires some algebraic preliminaries. We do not develop them in this article. We refer to [PM3] for a complete treatment. We content to prove here the following:

**Theorem 2.** *We consider a family  $(f_t)_{t \in \mathbf{C}^m}$  as in theorem 1 but we allow  $A \in GL_n(\mathbf{C})$  to be resonant. We are also given a family of formal linearizations  $(h_t)_{t \in \mathbf{C}^m}$  whose coefficients depend polynomially on  $t$ . We assume that the monomial of valuation  $l$  has as coefficient a polynomial of degree bounded above by  $C_0 + C_1 l$  for some  $C_0, C_1 > 0$ .*

*We have the following dichotomy:*

- 1) *The family  $(f_t)_{t \in \mathbf{C}^m}$  is holomorphically linearizable by the family  $(h_t)_{t \in \mathbf{C}^m}$ .*
- 2) *For all  $t \in \mathbf{C}^m$  except for an exceptional set  $E$  of  $\Gamma$ -capacity 0,  $h_t$  is diverging.*

One can also prove a statement similar to theorem 2 when  $(f_t)$  is not formally linearizable but the family  $(h_t)$  conjugates the family to a formal normal form ([PM3]). A particular relevant case is the one of a symplectic holomorphic diffeomorphism with an elliptic fixed point. The formal conjugacy to Birkhoff's normal form is then in general diverging (see [Si-Mo] section 30). The formal normal form situation is also relevant when  $A$  is not invertible.

### Central manifolds.

In situations where the dynamics is not linearizable, one can still have invariant manifolds through the fixed point (see for example [Pos], and [St] for a general treatment in the case of holomorphic vector fields). Usually one has a formal equation whose coefficients

depend polynomially on the coefficients of  $f_t$  thus on  $t$ . In these situations the following theorem applies.

**Theorem 3.** *Under the same assumptions as in theorem 1, we assume the existence of a formal invariant submanifold through 0 with equation*

$$F_t(z) = 0$$

with  $F_t : \mathbf{C}^n \rightarrow \mathbf{C}^p$  a formal mapping whose coefficients depend polynomially on  $t \in \mathbf{C}^m$ . More precisely, the coefficient of the monomial of valuation  $l$  is a polynomial on  $t$  of degree less than  $C_0 + C_1 l$  where  $C_0, C_1 > 0$  are constants.

We have the dichotomy:

- (1)  $F_t$  converges and defines an invariant submanifold for all  $t \in \mathbf{C}^m$ .
- (2) Except for an exceptional set of  $\Gamma$ -capacity 0 of parameter values  $t \in \mathbf{C}^m$ ,  $F_t$  diverges.

We have the same theorem for holomorphic vector fields. To be more specific, consider the situation treated by L.Stolovitch [Sto], for  $1 \leq j \leq n$ ,

$$\dot{z}_j = \lambda_j z_j + \sum_{i=1}^d t^i f_{j,i}(z)$$

where  $f_{j,i} = \mathcal{O}(2)$ . We assume that the linear part (which does not depend on  $t$ ) is in the Siegel domain, that is 0 belongs to the convex hull of  $\{\lambda_1, \dots, \lambda_n\}$ . We assume that the linear part is resonant, and the resonances,  $n_1, \dots, n_n \geq 1$  and any  $1 \leq j \leq n$ ,

$$\sum_{i=1}^n n_i \lambda_i - \lambda_j \neq 0 .$$

are generated by a finite number of resonances,  $1 \leq j \leq l$ ,  $r_j = (r_1, \dots, r_n) \neq 0$ ,  $r_j \in \mathbf{N}^n$ ,

$$(r_j, \lambda) = 0 .$$

Then there exists a formal change of variables  $w = h_t(z)$  with  $h_t(0) = 0$  and  $D_0 h_t = I$  which transforms the system into

$$\dot{w}_i = \lambda_i w_i + g_{i,t}(w)$$

with  $g_{i,t}(w) = \sum_{j=1}^l g_{i,j,t} y^{r_j}$ , and if  $\|r_j\| = 1$  then  $g_{i,j,t}(0) = 0$ . As constructed in [Sto], the coefficients of the formal normalization do depend polynomially on  $t$ .

**Theorem 4.** *With the previous assumption, we have the following dichotomy,*

- 1) For all value of  $t \in \mathbf{C}^m$  the formal normalization  $h_t$  converges, thus the submanifold  $\{w^{r_1} = 0, \dots, w^{r_n} = 0\}$  is invariant.
- 2) Except for an exceptional set of values of  $t$  of  $\Gamma$ -capacity 0, the normalization mappings  $h_t$  diverge.

According to [Sto], and assuming that the higher dimensional resonant Bruno condition on  $(\lambda_1, \dots, \lambda_n)$  holds, we are always in case (1).

### Singularities of holomorphic vector fields.

We consider a polynomial family of germs of holomorphic vector fields as before. But we assume here that the linear part is non-resonant, that is, for any  $n_1, \dots, n_2 \geq 1$  and any  $1 \leq j \leq n$ ,

$$\sum_{i=1}^n n_i \lambda_i - \lambda_j \neq 0 .$$

**Theorem 5.** *Under the above hypothesis, we have the dichotomy*

- 1) *The family of holomorphic vector fields is linearizable for all  $t$ .*
- 2) *Except for an exceptional set of values of  $t$  of  $\Gamma$ -capacity 0, the holomorphic vector fields are non-linearizable.*

In the case  $n = 2$  one has a complete correspondence of the problem of linearization of holomorphic vector fields as above and the problem of linearization of germs of holomorphic diffeomorphisms of  $(\mathbf{C}, 0)$  (see [PM-Yo] and the references there in). Yoccoz and the author proved that Bruno condition is optimal for the problem of linearization.

### Centralizers.

We discuss here the situation of one complex variable. The analysis generalizes similarly to higher dimension.

The study of centralizers of holomorphic germs generalizes the problem of linearization. We refer to [PM2] for proofs and references. In the group of holomorphic diffeomorphisms  $G = (\text{Diff}(\mathbf{C}, 0), \circ)$ , composed by holomorphic germs  $f$  with  $f(0) = 0$  and  $f'(0) \neq 0$ , we consider the centralizer of  $f$ ,

$$\text{Cent}(f) = \{g \in \text{Diff}(\mathbf{C}, 0); g \circ f = f \circ g\}$$

This group can be interpreted as the group of symmetries of  $f$  (i.e. those changes of variables conjugating  $f$  to itself). We have the following cases:

- 1) For germs with attracting or repelling fixed point at 0, i.e.  $f'(0) = e^{2\pi i \alpha}$  with  $\alpha \notin \mathbf{R}$ , the centralizer is a complex flow of dimension 1.
- 2) For germs with indifferent rational fixed point at 0, i.e.  $f'(0) = e^{2\pi i \alpha}$  with  $\alpha \in \mathbf{Q}$ , the centralizer is generated by root (for composition) of the germ (then it is discrete), or it is a one dimensional complex flow.

These cases are well understood. We discuss the last case in what follows.

- 3) For germs with an indifferent irrational fixed point at 0,  $f'(0) = e^{2\pi i \alpha}$  with  $\alpha \in \mathbf{R} - \mathbf{Q}$ , the centralizer can be a one-dimensional real flow (the linearizable case), discrete or uncountable. The occurrence of the last possibility was only proved recently in [PM2].

In this case centralizer is abelian and isomorphic to a subgroup of the circle  $\mathbf{T} = \mathbf{R}/\mathbf{Z}$  by the rotation number morphism,

$$\rho : \begin{array}{ccc} G & \longrightarrow & \mathbf{T} \\ f & \longmapsto & \log f'(0) \end{array}$$

We denote

$$G(f) = \rho(\text{Cent}(f)) .$$

Note that  $\mathbf{Z}\alpha \subset G(f)$ . The holomorphic germ  $f$  is linearizable if and only if the centralizer is full  $G(f) = \mathbf{T}$ , otherwise it is an  $F_\sigma$  and dense set of  $\mathbf{T}$  with 0 measure (and indeed 0 capacity). Moreover, all elements  $g \in \text{Cent}(f)$  are non-linearizable.

Thus how small is  $G(f)$  can be thought as a measure of how far is  $f$  from being linearizable. Thus the study of centralizers (apart from the motivation coming from the theory of foliations, see [PM2]) is motivated as a finer study of linearization. The question of determining if  $\beta \in G(f)$  is intimately connected with the common rational approximations of  $\alpha$  and  $\beta$ , as the following theorem of J. Moser shows ([Mo]). Let  $f$  be non-linearizable. If there exists  $\gamma, \tau > 0$  such that for any  $p \geq 1, q \in \mathbf{Z}$ ,

$$\min(|q\alpha - p_1|, |q\beta - p_2|) \geq \frac{\gamma}{q^\tau}$$

then  $\beta \notin G(f)$ . The necessity of an arithmetic condition in Moser's theorem is proved in [PM2].

We have:

**Theorem 6.** *Let  $f_t$  be a family of holomorphic germs as in theorem 1, with fixed linear part  $f'(0) = e^{2\pi i\alpha}$ ,  $\alpha \in \mathbf{R} - \mathbf{Q}$ . For any  $\beta \in \mathbf{T}$ , we have the following dichotomy:*

- 1) *For all  $t \in \mathbf{C}$ ,  $\beta \in G(f_t)$ .*
- 2) *Except for an exceptional polar set  $E \subset \mathbf{C}$ ,  $\beta \notin G(f_t)$ .*

### Further applications.

A complete treatment for the problem of linearization of resonant holomorphic germs is given in [PM3]. These techniques also apply to analytic K.A.M. of persistence of invariant tori. In [PM4] we study the Lindsted series for the standard map. Behind the technique used here there is an abstract theorem on holomorphic extension of Rothstein type for a certain type of power series. We discuss it in [PM4].



## 1) Proof of theorem 1.

### a) Nadirashvili and Bernstein lemmata.

For the definition of Green function, polar sets and other notions in potential theory we refer the reader to [Ra] for example (for a more encyclopedic treatment see [Tsu]).

Y. Ilyashenko in his article [Il] makes use of the following lemma attributed to N.S. Nadirashvili ([Na]).

**Lemma (Nadirashvili).** *Let  $E \subset \mathbf{C}$  be a compact set with positive measure in the disk  $\mathbf{D}_R$  of center 0 and radius  $R > 0$ . Let  $P$  be a polynomial of degree  $n$  such that for some  $M > 0$*

$$\|P\|_{C^0(E)} \leq M^n .$$

*Then there exists a constant  $C$  only depending on the measure of  $E$  and  $R > 0$  such that*

$$\|P\|_{\mathbf{D}_R} \leq C^n M^n .$$

We improve on [Il] observing that the measure of  $E$  is not the relevant quantity. Nadirashvili's lemma is a direct corollary of the classical Bernstein lemma in approximation theory and classical potential theory (see [Ra] p.156) and the fact that a set of positive measure is non-polar :

**Lemma (Bernstein).** *Let  $E \subset \mathbf{C}$  be a non-polar compact set (i.e.  $\text{cap}(E) > 0$ ). Let  $\Omega$  be the connected component of  $\overline{\mathbf{C}} - E$  containing  $\infty$ . Then for any polynomial  $P$  of degree  $n$ , we have for  $t \in \mathbf{C}$ ,*

$$|P(t)| \leq e^{ng_\Omega(t, \infty)} \|P\|_{C^0(K)}$$

where  $g_\Omega$  denotes the Green function of  $\Omega$ .

The proof is quite simple.

**Proof.** We can assume the polynomial monic. Then

$$u(t) = \log P(t) - \log \|P\|_{C^0(K)} - g_\Omega(t, \infty)$$

is sub-harmonic, is negative near  $\infty$  (because  $g_\Omega(t, \infty) = \log |t| + \text{cap}(E) + o(1)$ ), and  $\limsup u(t) \leq 0$  when  $t \rightarrow K$ . The maximum principle concludes the proof.  $\diamond$

For future reference we recall here that a countable union of polar sets is polar.

### b) $\Gamma$ -capacity.

We recall the definition of  $\Gamma$ -capacity and we refer to [Ro] for more properties. Let  $E \subset \mathbf{C}^m$ . The  $\Gamma$ -projection of  $E$  on  $\mathbf{C}^{m-1}$  is the set  $\Gamma_m^{m-1}(E)$  of  $z = (z_1, \dots, z_{m-1}) \in \mathbf{C}^{m-1}$  such that

$$E \cap \{(z, w) \in \mathbf{C}^m\}$$

has positive capacity in the complex plane  $\mathbf{C}_z = \{(z, w) \in \mathbf{C}^m\}$ . We define

$$\Gamma_m^1(E) = \Gamma_2^1 \circ \Gamma_3^2 \circ \dots \Gamma_m^{m-1}(E) .$$

Finally, the  $\Gamma$ -capacity is defined as

$$\Gamma\text{-Cap}(E) = \sup_{A \in U(m, \mathbf{C})} \text{Cap } \Gamma_m^1(A(E)) .$$

where  $A$  runs over all unitary transformations of  $\mathbf{C}^m$ .

Using the definition of  $\Gamma$ -capacity it is easy to see that we are reduced to prove the theorems for  $m = 1$

**c) Proof of theorem 1.**

We have the following elementary lemma.

**Lemma 1.1.** *The coefficient vectors  $h_i(t)$  of the formal linearization*

$$h_t(z) = z + \sum_{\substack{i=(i_1, \dots, i_n) \\ i_1 + \dots + i_n \geq 2}} h_i(t) z^i$$

*have coordinates that are polynomials in the parameter  $t = (t_1, \dots, t_m)$  of degree less than  $d(i_1 + \dots + i_n)$ .*

**Proof.** We can assume that  $A$  is in upper triangular Jordan normal form. We solve the functional equation

$$A \circ h_t = h_t \circ f_t$$

identifying coordinates and developing in homogeneous vector monomials. By induction on  $|i| = i_1 + \dots + i_n$  we do determine successively the vectors  $h_i(t)$  that depend on coefficients of  $f_t$  and lower order  $h_j(t)$ 's,  $|j| < |i|$ . By induction, the linear equations determining  $h_i(t)$  do have the form

$$(A - M_i)h_i(t) = \sum_{|j| < |i|} c_j(t)h_j(t)$$

where the matrix  $M_i$  is upper triangular, only depends on  $A$  and  $i$  (but not on the parameter  $t$ ), has diagonal coefficients products of eigenvalues of  $A$  (thus  $A - M_i$  is invertible) and in the left hand side the coefficients  $c_j(t)$  are polynomials in  $t$  of total degree at most  $d$ . By induction the result follows.  $\diamond$

**Proof of theorem 1.** According to previous section we are reduced to prove the theorem to the case  $m = 1$ . Let

$$E = \{t \in \mathbf{C}; f_t \text{ is linearizable} \} .$$

We want to show that  $E$  is polar or the whole complex plane. We have

$$E = \bigcup_{j \geq 1} E_j$$

where  $E_j$  the set of parameters  $t$  such that  $h_t$  has radius of convergence larger or equal to  $1/j$ . Thus if  $E$  is non-polar, we have that for some  $j \geq 1$ ,  $E_j$  is not polar. Thus there exists  $\rho_0 > 0$  such that for all  $t \in E_j$ ,

$$\varphi(t) = \limsup_{|i| \rightarrow +\infty} \|h_i(t)\| \rho_0^{-|i|} < +\infty .$$

The function  $\varphi$  is lower semicontinuous, and

$$E_j = \bigcup_p L_p$$

where  $L_p = \{z \in E_j; \varphi(t) \leq p\}$  is closed. By Baire theorem for some  $p$ ,  $L_p$  has non-empty interior (with respect to  $E_j$ ), thus this  $L_p$  has positive capacity. Finally we found a compact set  $C = L_p$  of positive capacity such that there exists  $\rho_1 > 0$  such that for any  $t \in C$  and all  $i \in \mathbf{N}^n$ ,

$$\|h_i(t)\|_{C^0(C)} \leq \rho_1^{|i|}.$$

Using Bernstein's lemma and lemma 1.1 we get that for any compact set  $K \subset \mathbf{C}$  we have

$$\|h_i(t)\|_{C^0(K)} \leq C(K)^{d|i|} \rho_1^{|i|},$$

for some constant  $C(K)$  depending only on  $K$ . Thus  $f_t$  is linearizable for any  $t \in \mathbf{C}$ . The constant  $C(K)$  can be estimated by the precise form of Bernstein lemma as

$$C(K) = \exp(\sup_K g_\Omega(t, \infty))$$

where  $\Omega$  is the connected component containing  $\infty$  of the complement of  $C$ . The asymptotic

$$g_\Omega(t, \infty) \approx \log |t|$$

for  $t \rightarrow \infty$  gives the lower estimate on the radius of convergence.  $\diamond$

### c) Proof of the corollaries.

Corollary 1 is just a particular case of theorem 1. Corollary 3 part (1) also. Now using corollary 3 part (1) we prove

**Corollary (Yoccoz, [Yo]).** *The quadratic polynomial  $P_\alpha(z) = e^{2\pi i \alpha} z + z^2$  is non-linearizable when  $\alpha$  is not a Bruno number.*

**Proof.** For this, pick  $f$  non-linearizable (that exists from [Yo]) and consider  $f_t(z) = tP_\alpha(z) + (1-t)f(z)$ . Since  $f_1$  is not linearizable, all  $f_t$  except for a polar set  $E$  of values of  $t$  are not linearizable. By Douady-Hubbard straitening theorem  $\mathbf{C} - E$  is a neighborhood of 0 and 0 is not linearizable.  $\diamond$

Now by the same argument, part (2) and (3) of corollary 3 follow.

Now we prove:

**Corollary (Pérez-Marco, [PM1]).** *If  $P$  is a structurally stable polynomial in the space*

$$\mathcal{P}_{\lambda,d} = \{P(z) = \lambda z + a_2 z^2 + \dots + a_d z^d; (a_2, \dots, a_d) \in \mathbf{C}^{d-1}\}$$

*then  $P$  is not linearizable.*

**Proof.** Just consider

$$f_t(z) = tP(z) + (1-t)P_\alpha(z)$$

and do the same proof.  $\diamond$

Now corollary 2 follows from part (1) of corollary 3.

Corollary 4 is not a strict corollary of theorem 1 but exactly the same proof applies, observing that the coefficients of the linearization are polynomial functions of the coefficients of the rational function with appropriate degree.

For corollary 5 only the last assertion is not immediate. If one of the eigenvalues of  $A$  violates Bruno condition,  $\lambda_1$  for example, then

$$(z_1, \dots, z_n) \mapsto (\lambda_1 z_1 + z_1^2, \lambda_2 z_2, \dots, \lambda_n z_n)$$

is not linearizable, thus the first part applies giving a rich family of polynomial non-linearizable examples.

## 2) Proof of the other theorems.

### a) Formal linearizations and theorem 2.

The proof of theorem 2 is similar to the proof given in the previous section of theorem 1. For a complete study of the resonant case we refer to [PM3]. We just mention here the new difficulties that appear.

Assume that we have a germ of holomorphic diffeomorphism

$$f : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$$

with resonant linear part  $A = D_0 f \in GL_n(\mathbf{C})$ .

The formal linearization  $h$  is not always unique when the linear part  $A$  is resonant or not invertible. For example, for  $n = 2$  and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then if  $h$  is a formal linearization then  $l \circ h$  is also one, where

$$l(z_1, z_2) = (z_1 + k(z_2), z_2) .$$

One can show that all linearizations can be obtained that way ([PM3]). If we consider a polynomial family  $(f_t)_{t \in \mathbf{C}^m}$ , to request that  $h_t$  has coefficients depending polynomially on  $t$  does not improve things. One then can take various  $k_t$  depending polynomially on  $t$ . Thus the family  $(h_t)$  with this further restriction is not unique.

This presents a subtle problem in order to prove the non-linearizability. Considering a polynomial parameter family of formal linearizations of a fixed map  $f$ ,  $(h_t)$ , we may be in the second case, but this does not mean that  $f$  is not linearizable. For instance, if the exceptional set  $E$  is not empty, then  $f$  will be linearizable ! The question of non-linearizability is harder to answer. In [PM3] we show that if the polynomial family of linearizations is chosen in a natural way, this difficulty does not arise.

## b) Other theorems.

The proofs are similar than in section 1. We just comment on the particularities of each problem.

For an explicit example where theorem 3 applies one can workout the example of J. Poschel [Pos]. The polynomial dependence with the appropriate bound on the degrees follows from the formal computation of the formal equation of the invariant manifold.

Theorem 4 is proved in a similar way. We refer to [Sto] for the formal computation of a normalizing map with polynomial dependence on the parameter  $t$  with the appropriate degrees. One can workout in this situation similar results than in [PM3].

The linearization in theorem 5 is unique and it is well known ([Ar]) that it depends polynomially on  $t$  with the appropriate degrees. Thus the same proof applies. Note that in  $\mathbf{C}^2$ , by [PM-Y] one can realize any germ of holomorphic diffeomorphism in  $(\mathbf{C}, 0)$  as holonomy of a singularity of holomorphic vector field of the type considered. The realization of the quadratic map is also structurally stable in the following sense: Any nearby holomorphic vector field has a holonomy that is quasi-conformally conjugated to the quadratic polynomial. Thus for any one parameter polynomial family containing this vector field, if  $\alpha$  is not a Bruno number, we will be in case 2.

For the proof of theorem 6, we give the induction formulas for the coefficients of  $g_\beta$ . Let  $\mu = e^{2\pi i\alpha} = f_1$  and  $\lambda = e^{2\pi i\alpha} = g_1$ , and

$$f(z) = \sum_{n=1}^{+\infty} f_n z^n$$

$$g(z) = \sum_{n=1}^{+\infty} g_n z^n$$

Identifying terms of degree  $n \geq 2$  in the equation  $g \circ f = f \circ g$ , we get for  $n \geq 2$ ,

$$g_n = \frac{\mu^n - \mu}{\lambda^n - \lambda} f_n + \sum_{p=1}^{+\infty} f_p \sum_{\substack{i_1 + \dots + i_p = n \\ i_j \geq 1}} g_{i_1} \dots g_{i_p} + \sum_{p=1}^{+\infty} g_p \sum_{\substack{i_1 + \dots + i_p = n \\ i_j \geq 1}} f_{i_1} \dots f_{i_p}$$

And by induction the coefficients of  $g_\beta$  depend polynomially on  $t$  and have the appropriate degrees.

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